Matchings and Path Covers with applications to Domination in Graphs

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Abstract

Let G be a graph with no isolated vertex. A matching in G is a set of edges that are pairwise not adjacent in G, while the matching number, $\alpha'(G)$, of G is the maximum size of a matching in G. The path covering number, $\mathrm{pc}(G)$, of G is the minimum number of vertex disjoint paths such that every vertex belongs to a path in the cover. We show that if G has order n, then $\alpha'(G) + \frac{1}{2}\mathrm{pc}(G) \ge \frac{n}{2}$ and we provide a constructive characterization of the graphs achieving equality in this bound. It is known that $\gamma(G) \le \alpha'(G)$ and $\gamma_t(G) \le \alpha'(G) + \mathrm{pc}(G)$, where $\gamma(G)$ and $\gamma_t(G)$ denote the domination and the total domination number of G. As an application of our result on the matching and path cover numbers, we show that if G is a graph with $\delta(G) \ge 3$, then $\gamma_t(G) \le \alpha'(G) + \frac{1}{2}(\mathrm{pc}(G) - 1)$, and this bound is tight. A set S of vertices in G is a neighborhood total dominating set of G if it is a dominating set of G with the property that the subgraph induced by the open neighborhood of the set S has no isolated vertex. The neighborhood total domination number, $\gamma_{\mathrm{nt}}(G)$, is the minimum cardinality of a neighborhood total dominating set of G. We observe that $\gamma(G) \le \gamma_{\mathrm{nt}}(G) \le \gamma_t(G)$. As a further application of our result on the matching and path cover numbers, we show that if G is a connected graph on at least six vertices, then $\gamma_{\mathrm{nt}}(G) \le \alpha'(G) + \frac{1}{2}\mathrm{pc}(G)$ and this bound is tight.

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1 Introduction

Two distinct edges in a graph G are *independent* if they are not adjacent in G. A *matching* in G is a set of (pairwise) independent edges, while a matching of maximum cardinality is a *maximum matching*. The number of edges in a maximum matching of a graph G is called the *matching number* of G, denoted by $\alpha'(G)$. Let M be a specified matching in a graph G. A vertex v of G is an *M-matched vertex* if v is incident with an edge of M; otherwise, v is an *M-unmatched vertex*. An *M-alternating path* of G is a path whose edges are alternately in M and not in M. An *M-augmenting path* is an M-alternating path that begins and ends with M-unmatched vertices. Given an M-augmenting path P, we denote by $M \triangle E(P)$ the symmetric difference of M and $E(P)$, i.e., $M \triangle E(P) = (M \setminus E(P)) \cup (E(P) \setminus M)$. Matchings in graphs are extensively studied in the literature (see, for example, the excellent survey articles by Plummer [23] and Pulleyblank [24]).

A *path cover* in a graph G is a collection of vertex disjoint paths such that every vertex belongs to exactly one path. The cardinality of a minimum path cover is the *path covering number* of G which we denote by $\mathrm{pc}(G)$.

A *dominating set* in a graph G is a set S of vertices of G such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A *total dominating set*, abbreviated a TD-set, of a graph G with no isolated vertex is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [10, 11]. Total domination is now well studied in graph theory. For a recent book on the topic, see [20]. A survey of total domination in graphs can also be found in [12].

Arumugam and Sivagnanam [2] introduced and studied the concept of neighborhood total domination in graphs. A *neighbor* of a vertex v is a vertex different from v that is adjacent to v. The *neighborhood of a set* S is the set of all neighbors of vertices in S. A *neighborhood total dominating set*, abbreviated NTD-set, in a graph G is a dominating set S in G with the property that the subgraph induced by the open neighborhood of the set S has no isolated vertex. The *neighborhood total domination number* of G, denoted by $\gamma_{\mathrm{nt}}(G)$, is the minimum cardinality of a NTD-set of G. Since every TD-set is a NTD-set, and since every NTD-set is a dominating set, we have the following observation first observed by Arumugam and Sivagnanam in [2].

Observation 1 ([2]) *If G is a graph with no isolated vertex, then $\gamma(G) \le \gamma_{\mathrm{nt}}(G) \le \gamma_t(G)$.*

By Observation 1, the neighborhood total domination number is squeezed between arguably the two most important domination parameters, namely the domination number and the total domination number.

1.1 Terminology and Notation

For notation and graph theory terminology not defined herein, we refer the reader to [10]. Let G be a graph with vertex set $V(G)$ of order $n = |V(G)|$ and edge set $E(G)$ of size $m = |E(G)|$, and let v be a vertex in V. We denote the *degree* of v in G by $d_G(v)$. The minimum degree among the vertices of G is denoted by $\delta(G)$. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$.

A *cycle* and *path* on n vertices are denoted by C_n and P_n, respectively. A *star* on $n \geq 2$ vertices is a tree with a vertex of degree $n-1$ and is denoted by $K_{1,n-1}$. A *double star* is a tree containing exactly two vertices that are not leaves (which are necessarily adjacent). A *subdivided star* is a graph obtained from a star on at least two vertices by subdividing each edge exactly once. We note that the smallest two subdivided stars are the paths P_3 and P_5.

The *open neighborhood* of v is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, we simply write $\mathrm{d}(v)$, $N(v)$, $N[v]$, $N(S)$ and $N[S]$ rather than $d_G(v)$, $N_G(v)$, $N_G[v]$, $N_G(S)$ and $N_G[S]$, respectively. As observed in [14] a NTD-set in G is a set S of vertices such that $N[S] = V$ and $G[N(S)]$ contains no isolated vertex.

A *rooted tree* distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex u such that the unique (r, u)-path contains v. Let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and let $D[v] = D(v) \cup \{v\}$. A *non-leaf* of a tree T is a vertex of T of degree at least 2 in T.

1.2 Known Results

Bounds relating the domination number and the matching number are studied, for example, in [3, 7]. As a consequence of a result due to Bollobás and Cockayne [3], the domination number of every graph with no isolated vertex is bounded above by its matching number.

Theorem 2 ([3]) *For every graph G with no isolated vertex, $\gamma(G) \leq \alpha'(G)$.*

The total domination number versus the matching number in a graph has been studied in several papers (see, for example, [8, 13, 16, 18, 19, 21, 22, 25, 26] and elsewhere). Unlike the domination number, the total domination number and the matching number of a graph are generally incomparable, even for arbitrarily large, but fixed (with respect to the order of the graph), minimum degree as shown in [13]. The following upper bound on the total domination in terms of its matching number and path covering number is presented in [8].

Theorem 3 ([8]) *For every graph G with no isolated vertex, $\gamma_t(G) \leq \alpha'(G) + \mathrm{pc}(G)$, and this bound is tight.*

The following upper bound on the neighborhood total domination number of a connected graph in terms of its order is established in [14].

Theorem 4 ([14]) *Let G be a connected graph of order $n \geq 3$. Then, $\gamma_{\rm nt}(G) \leq (n+1)/2$ with equality if and only if $G = C_5$ or G is a subdivided star.*

1.3 The Family $\mathcal{T}$

By a *weak partition* of a set we mean a partition of the set in which some of the subsets may be empty. For our purposes we define a *labeling* of a tree T as a weak partition $S = (S_A, S_B, S_C)$ of $V(T)$. We will refer to the pair (T, S) as a *labeled tree.* The *label* or *status* of a vertex v, denoted ${\rm sta}(v)$, is the letter $x \in \{A, B, C\}$ such that $v \in S_x$. We now define the following family of trees.

Let $\mathcal{T}$ be the family of labeled trees that: (i) contains (P_3, S_0^*) where S_0^* is the labeling that assigns status A to the central vertex of P_3 and status B to the two leaves; and (ii) is closed under the four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ that are listed below, which extend the tree T' to a tree T by attaching a tree to the vertex $v \in V(T')$, called the *attacher* of T'.

- **Operation $\mathcal{O}_1$.** Let v be a vertex with ${\rm sta}(v) = A$. Add a vertex u_1 and the edge vu_1. Let ${\rm sta}(u_1) = B$.

- **Operation $\mathcal{O}_2$.** Let v be a vertex with ${\rm sta}(v) = B$. Add a path u_1u_2 and the edge vu_1. Change the status of v from B to C, and so ${\rm sta}(v) = C$, and let ${\rm sta}(u_1) = A$ and ${\rm sta}(u_2) = B$.

- **Operation $\mathcal{O}_3$.** Let v be an arbitrary vertex of T'. Add a path $u_1u_2u_3$ and the edge vu_2. Let ${\rm sta}(u_1) = {\rm sta}(u_3) = B$ and ${\rm sta}(u_2) = A$. Further, if ${\rm sta}(v) = B$, then change the status of v from B to C.

- **Operation $\mathcal{O}_4$.** Let v be a vertex with ${\rm sta}(v) = A$. Add a path $u_1u_2u_3u_4u_5$ and the edge vu_3. Let ${\rm sta}(u_1) = {\rm sta}(u_5) = B$, ${\rm sta}(u_2) = {\rm sta}(u_4) = A$, and ${\rm sta}(u_3) = C$.

The four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ are illustrated in Figure 1.

We shall need the following properties of labeled trees (T, S) in the family $\mathcal{T}$ that follow immediately from the way in which trees in the family $\mathcal{T}$ are constructed.

Observation 5 *If $(T, S) \in \mathcal{T}$ is a labeled tree for some labeling S, then T has the following properties:*
(a) *Every support vertex of T belongs to S_A.*
(b) *The set S_B is the set of leaves of T.*
(c) *Every vertex in S_A has at least two neighbors in $S_B \cup S_C$.*
(d) *If $v \in S_C$, then $N(v) \subseteq S_A$.*

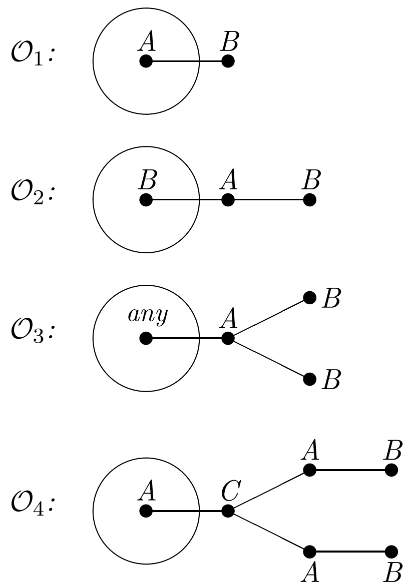

Figure 1: The four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$.

2 Main Results

We have two immediate aims in this paper. Our first aim is to establish a lower bound relating the matching number and the path covering number of a graph in terms of the order of the graph. Our second aim is to establish an upper bound on the neighborhood total domination number of a graph in terms of its matching and path covering numbers. More precisely, we shall prove the following results, proofs of which are presented in Sections 3 and 4.

Theorem 6 *Let T be a tree of order $n \geq 3$. Then, $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) \geq \frac{n}{2}$ with equality if and only if $(T, S) \in \mathcal{T}$ for some labeling S.*

Theorem 7 *If G is a graph of order n, then $\alpha'(G) + \frac{1}{2}\mathrm{pc}(G) \geq \frac{n}{2}$. Further for $n \geq 3$, $\alpha'(G) + \frac{1}{2}\mathrm{pc}(G) = \frac{n}{2}$ if and only if G has a spanning tree T such that*

(a) *$(T, S) \in \mathcal{T}$ for some labeling S.*
(b) *$\alpha'(G) = \alpha'(T)$.*
(c) *$\mathrm{pc}(G) = \mathrm{pc}(T)$.*

As an application of Theorem 7, we show that the bound of Theorem 3 can be improved considerably if we restrict the minimum degree of the graph to be at least three.

Corollary 8 *If G is a graph with $\delta(G) \geq 3$, then $\gamma_t(G) \leq \alpha'(G) + \frac{1}{2}(\mathrm{pc}(G) - 1)$, and this bound is tight.*

As a further application of Theorem 7, we establish the following upper bound on the neighborhood total domination number of a graph in terms of its matching and path covering numbers.

Corollary 9 *If G is a connected graph of order at least 3, then*

$$\gamma_{\rm nt}(G) \le \alpha'(G) + \frac{1}{2}{\rm pc}(G)$$

unless $G \in \{P_3, P_5, C_5\}$ in which case $\gamma_{\rm nt}(G) = \alpha'(G) + \frac{1}{2}({\rm pc}(G) + 1)$.

3 Proof of Theorem 6 and Theorem 7

In this section, we establish a lower bound for $\alpha'(G) + \frac{1}{2}{\rm pc}(G)$. For this purpose, we shall need the following results on matchings and path covers in trees.

Lemma 10 *If T is a tree of order $n \ge 1$, then $\alpha'(T) + \frac{1}{2}{\rm pc}(T) \ge \frac{n}{2}$.*

Proof. We proceed by induction on the order $n \ge 1$ of a tree T. If $n \in \{1, 2, 3\}$, then $T = P_n$ and the result follows readily. This establishes the base case. Let $n \ge 4$ and assume that if T' is a tree of order n' where $n' < n$, then $\alpha'(T') + \frac{1}{2}{\rm pc}(T') \ge \frac{n'}{2}$. Let T be a tree of order n. If T is a star, then $\alpha'(T) = 1$ and ${\rm pc}(T) = n - 2$, and so $\alpha'(T) + \frac{1}{2}{\rm pc}(T) = \frac{n}{2}$. Hence we may assume that ${\rm diam}(T) \ge 3$, for otherwise the desired result follows. Let P be a longest path in T and suppose that P is an (r, u)-path. Necessarily, r and u are leaves in T. We now root the tree T at the vertex r. Let v be the parent of u, and let w be the parent of v in the rooted tree T.

Let $\mathcal{P}$ be a minimum path cover in T and let P_v be the path in $\mathcal{P}$ containing v. Thus, $|\mathcal{P}| = {\rm pc}(T)$. By the minimality of the path cover $\mathcal{P}$ in T, we may assume, renaming the children of v if necessary, that $u \in V(P_v)$.

Suppose that $d_T(v) = 2$. In this case, let $T' = T - \{u, v\}$ and let T' have order n', and so $n' = n - 2 \ge 2$. Applying the inductive hypothesis to T', we note that $\alpha'(T') + \frac{1}{2}{\rm pc}(T') \ge \frac{n'}{2}$. Every matching in T' can be extended to a matching in T by adding to it the edge uv, implying that $\alpha'(T) \ge \alpha'(T') + 1$. By our earlier assumption, the vertex $u \in V(P_v)$. If $w \in V(P_v)$, then replacing the path P_v in $\mathcal{P}$ with the path $P_v - \{u, v\}$ produces a path cover in T' of size $|\mathcal{P}|$. If $w \notin V(P_v)$, then removing the path P_v from $\mathcal{P}$ produces a path cover in T' of size $|\mathcal{P}| - 1$. In both cases, we produce a path cover in T' of size at most $|\mathcal{P}|$, implying that ${\rm pc}(T') \le {\rm pc}(T)$. Therefore, $\alpha'(T) + \frac{1}{2}{\rm pc}(T) \ge \alpha'(T') + \frac{1}{2}{\rm pc}(T') + 1 \ge \frac{n'}{2} + 1 = \frac{n}{2}$. Hence we may assume that $d_T(v) \ge 3$, for otherwise the desired result holds.

Let $C(v) = \{u_1, u_2, \ldots, u_k\}$ denote the children of v, where $u = u_1$. By assumption, $k \ge 2$. We now let T' be the tree obtained from T by deleting v and all children of v; that is, $T' = T - D[v]$. Let T' have order n', and so $n' = n - k - 1$. Applying the inductive hypothesis to T', we note that $\alpha'(T') + \frac{1}{2}{\rm pc}(T') \ge \frac{n'}{2}$. Every matching in T' can be extended

to a matching in T by adding to it the edge uv, implying that $\alpha'(T) \ge \alpha'(T') + 1$. By the minimality of the path cover $\mathcal{P}$ in T, the path P_v in $\mathcal{P}$ contains either the vertex w or a child of v different from u_1. If $w \in V(P_v)$, then we can remove from $\mathcal{P}$ both the path P_v and the path in $\mathcal{P}$ that consists only of the vertex u_2 and replace these two paths with the path $P_v - \{u, v\}$ and the path u_1vu_2. Hence, we can choose $\mathcal{P}$ so that the path P_v in $\mathcal{P}$ that contains v is the path u_1vu_2. We note that if $k \ge 3$, then each child u_i of v belongs to a path in $\mathcal{P}$ that consists only of the vertex u_i. Removing the $k-1$ paths from $\mathcal{P}$ that contain vertices in $D[v]$, we produce a path cover in T' of size $|\mathcal{P}| - (k-1) = \mathrm{pc}(T) - k + 1$, implying that $\mathrm{pc}(T') \le \mathrm{pc}(T) - k + 1$. Hence,

$$\begin{array}{rcl} \alpha'(T) + \frac{1}{2}\mathrm{pc}(T) & \ge & (\alpha'(T') + 1) + \frac{1}{2}(\mathrm{pc}(T') + k - 1) \\ & = & (\alpha'(T') + \frac{1}{2}\mathrm{pc}(T')) + \frac{1}{2}(k+1) \\ & \ge & \frac{n'}{2} + \frac{k+1}{2} \\ & = & \frac{n-k-1}{2} + \frac{k+1}{2} \\ & = & \frac{n}{2}. \end{array}$$

This completes the proof of Lemma 10. $\Box$

Let $(T, S) \in \mathcal{T}$ be a labeled tree for some labeling S. Then there is a sequence of labeled trees $(T_0, S_0), (T_1, S_1), \ldots, (T_k, S_k)$ such that $(T_0, S_0) = (P_3, S_0^*)$, $(T_k, S_k) = (T, S)$, and if $k \ge 1$, then for $i \in \{1, \ldots, k\}$, the labeled tree (T_i, S_i) can be obtained from (T_{i-1}, S_{i-1}) by one of the operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ or $\mathcal{O}_4$. We call the number of terms in such a sequence of labeled trees that is used to construct (T, S), the *length* of the sequence. In particular, the above sequence has length k. We remark that a sequence of labeled trees used to construct (T, S) is not necessarily unique. Further, the length of such sequences may differ. We shall need the following properties of trees in the family $\mathcal{T}$.

Lemma 11 *If $(T, S) \in \mathcal{T}$ is a labeled tree for some labeling S, then T has the following properties.*

(a) *If $w \in S_A$ and M is an arbitrary maximum matching in T, then w is M-matched.*
(b) *If $w \in S_B \cup S_C$, then there exists a maximum matching M in T such that w is M-unmatched.*
(c) $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = \frac{n}{2}$.

Proof. Let $(T, S) \in \mathcal{T}$ be a labeled tree for some labeling S. We proceed by induction on the length, $k \ge 0$, of a sequence used to construct (T, S). If $k = 0$, then $T = P_3$ and $S = S_0^*$, and Properties (a), (b) and (c) hold. This establishes the base case. Let $k \ge 1$ and assume that if the length of the sequence used to construct a labeled tree $(T', S') \in \mathcal{T}$ is less than k, then Properties (a), (b) and (c) hold for the labeled tree (T', S'). Let $(T, S) \in \mathcal{T}$ and let $(T_0, S_0), (T_1, S_1), \ldots, (T_k, S_k)$ be a sequence of length k used to construct (T, S), where $(T_0, S_0) = (P_3, S_0^*)$ and $(T_k, S_k) = (T, S)$. Let $T' = T_{k-1}$ and let $S' = S_{k-1}$. Then, $(T', S') \in \mathcal{T}$. Let M be a maximum matching in T and let M' be the restriction of M to the tree T'; that is, $M' = M \cap E(T')$. We consider four cases, depending on the operation applied to (T', S') in order to obtain (T, S).

Case 1. (T, S) can be obtained from (T', S') by operation $\mathcal{O}_1$. Let v be the attacher in T' and let u_1 be the vertex added to T' to obtain T. Then, $\mathrm{sta}(v) = A$ and $\mathrm{sta}(u_1) = B$. We show that $\alpha'(T') = \alpha'(T)$. Since M is a maximum matching in T, the vertex v is M-matched. If $vu_1 \notin M$, then $M' = M$, implying that $\alpha'(T') \ge |M'| = |M| = \alpha'(T)$. If $vu_1 \in M$, then $M' = M \setminus \{vu_1\}$ and $|M'| = |M| - 1$. Applying the inductive hypothesis to the labeled tree $(T', S') \in \mathcal{T}$, the matching M' is not a maximum matching in T' since the vertex $v \in S'_A$ is M'-unmatched. Hence in this case, $\alpha'(T') \ge |M'| + 1 = |M| = \alpha'(T)$. In both cases, $\alpha'(T') \ge \alpha'(T)$. Conversely, every matching in T' is a matching in T, and so $\alpha'(T) \ge \alpha'(T')$. Consequently, $\alpha'(T') = \alpha'(T)$.

We show firstly that Property (a) holds. Let $w \in S_A$. Then, $w \in V(T')$. Let M be an arbitrary maximum matching in T and let M' be the restriction of M to the tree T'. Suppose, to the contrary, that w is M-unmatched. If $M = M'$, then applying the inductive hypothesis to the labeled tree $(T', S') \in \mathcal{T}$ the vertex w is M'-matched in T' and therefore M-matched in T, a contradiction. Hence, $M \neq M'$, implying that $vu_1 \in M$. Thus, $M' = M \setminus \{vu_1\}$ and $|M'| = |M| - 1$. We now consider the matching M'. Let P_v be a longest path in T' starting at v and whose edges are alternately not in M' and in M' and whose vertices are alternately in S'_A and in S'_C. Let P_v be a (v, z)-path. By Observation 5, every vertex in the set S'_A has at least two neighbors in $S'_B \cup S'_C$.

If $v = z$, then in the tree T' the vertex v has no neighbor in S'_C and therefore at least two neighbors in S'_B. Let u' be a neighbor of v in T' that belongs to S'_B. By Observation 5(b), the vertex u' is a leaf. Since $vu_1 \in M$, we note that u' is M-unmatched. We now consider the matching $M^* = M' \cup \{u'v\}$. Since M^* has size $|M'| + 1 = |M| = \alpha'(T) = \alpha'(T')$, the matching M^* is a maximum matching in T'. Since $w \notin \{u', v\}$ and w is M-unmatched, the vertex w is M^*-unmatched, contradicting Property (a). Hence, $v \neq z$.

If $z \in S'_C$, then by the maximality of the path P_v the vertex z is M'-unmatched. Thus, P_v is an M'-augmenting (v, z)-path in T'. We now consider the matching $M^* = M \triangle E(P_v)$. Since M^* has size $|M'| + 1 = |M| = \alpha'(T) = \alpha'(T')$, the matching M^* is a maximum matching in T'. Since $w \notin V(P_v)$ and w is M-unmatched, the vertex w is M^*-unmatched, contradicting Property (a). Hence, $z \in S'_A$.

Since $z \in S'_A$, the vertex z has at least two neighbors in $S'_B \cup S'_C$. By the maximality of the path P_v the vertex z has no neighbor in S'_C except for the vertex immediately preceding it on the path P_v (that is M-matched to z). Thus, z has a neighbor, z' say, that belongs to S'_B. By Observation 5(b), the vertex z' is a leaf. We now extend the path P_v by adding to it the vertex z' and the edge zz' to produce a new path P which is an M'-augmenting (v, z')-path in T'. We now consider the matching $M^* = M \triangle E(P)$. Since M^* has size $|M'| + 1 = |M| = \alpha'(T) = \alpha'(T')$, the matching M^* is a maximum matching in T'. Since $w \notin V(P)$ and w is M-unmatched, the vertex w is M^*-unmatched, contradicting Property (a). Therefore, w is M-matched. Since w was chosen to be an arbitrary vertex in S_A, this proves that Property (a) holds.

We show secondly that Property (b) holds. Let $w \in S_B \cup S_C$. Suppose that $w = u_1$. Let M^* be an arbitrary maximum matching in T'. Since $\alpha'(T') = \alpha'(T)$, the matching M^* is a maximum matching in T such that w is M^*-unmatched, as desired. Hence we may

assume that $w \neq u_1$, for otherwise the desired result holds. Thus, $w \in V(T')$. Applying the inductive hypothesis to the labeled tree (T', S'), there exists a maximum matching M_w in T' such that w is M_w-unmatched. Since $\alpha'(T') = \alpha'(T)$, the matching M_w is a maximum matching in T such that w is M_w-unmatched. This establishes Property (b).

We prove next that Property (c) holds. As observed earlier, $\alpha'(T') = \alpha'(T)$. Either $\mathrm{pc}(T') = \mathrm{pc}(T) - 1$ or $\mathrm{pc}(T') = \mathrm{pc}(T)$, implying by Lemma 10 that $\frac{n}{2} \leq \alpha'(T) + \frac{1}{2}\mathrm{pc}(T) \leq \alpha'(T') + \frac{1}{2}(\mathrm{pc}(T') + 1) = \frac{n'}{2} + \frac{1}{2} = \frac{n}{2}$. Hence we must have equality throughout this inequality chain. In particular, $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = \frac{n}{2}$. This establishes Property (c).

Case 2. (T, S) can be obtained from (T', S') by operation $\mathcal{O}_2$. Let v be the attacher in T' and let u_1u_2 be the path added to T' and vu_1 the added edge. Then, $\mathrm{sta}_{T'}(v) = B$ but $\mathrm{sta}_T(v) = C$, while $\mathrm{sta}(u_1) = A$ and $\mathrm{sta}(u_2) = B$. If $u_1u_2 \notin M$, then by the maximality of M, the edge $u_1v \in M$ and we can simply replace the edge u_1v in M with the edge u_1u_2. Hence we may choose M so that $u_1u_2 \in M$. Thus, $M' = M \setminus \{u_1u_2\}$ is a matching in T', implying that $\alpha'(T') \geq |M'| = |M| - 1 = \alpha'(T) - 1$. Every matching in T' can be extended to a matching in T by adding to it the edge u_1u_2, and so $\alpha'(T) \geq \alpha'(T') + 1$. Consequently, $\alpha'(T') = \alpha'(T) - 1$.

We show firstly that Property (a) holds. Let $w \in S_A$. Let M be an arbitrary maximum matching in T and let M' be the restriction of M to the tree T'. Suppose, to the contrary, that w is M-unmatched. If $w = u_1$, then by the maximality of M, the vertex w is M-matched, a contradiction. Hence, $w \in V(T')$. Suppose $vu_1 \in M$. Then, $M' = M \setminus \{vu_1\}$. Since $\alpha'(T') = \alpha'(T) - 1$ and M' has size $|M| - 1 = \alpha'(T) - 1 = \alpha'(T')$, the matching M' is a maximum matching in T'. Applying the inductive hypothesis to the labeled tree $(T', S') \in \mathcal{T}$ the vertex w is M'-matched in T' and therefore M-matched in T, a contradiction. Therefore, $vu_1 \notin M$, implying that $u_1u_2 \in M$ and $M' = M \setminus \{u_1u_2\}$. Once again, M' has size $\alpha'(T) - 1$ and is a maximum matching in T'. Applying the inductive hypothesis to the labeled tree $(T', S') \in \mathcal{T}$, the vertex w is M'-matched in T' and therefore M-matched in T, a contradiction. This establishes Property (a).

We show secondly that Property (b) holds. Let $w \in S_B \cup S_C$. Suppose that $w = u_2$. Applying the inductive hypothesis to the labeled tree (T', S'), there is a maximum matching M^* of T' such that the vertex v which has status B in (T', S') is M^*-unmatched. Thus the matching $M_w = M^* \cup \{vu_1\}$ is a maximum matching in T such that w is M_w-unmatched. Hence we may assume that $w \in V(T')$, for otherwise the desired result holds. (Possibly, $w = v$, in which case w has status B in (T', S') and status C in (T, S).) Applying the inductive hypothesis to the labeled tree $(T', S') \in \mathcal{T}$, there is a maximum matching M^* of T' such that the vertex w is M^*-unmatched. Thus the matching $M_w = M^* \cup \{u_1u_2\}$ is a maximum matching in T such that w is M_w-unmatched. This establishes Property (b).

We prove next that Property (c) holds. As observed earlier, $\alpha'(T') = \alpha'(T) - 1$. Since the attacher vertex v is a leaf in T', we note that $\mathrm{pc}(T) = \mathrm{pc}(T')$. Therefore, $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = (\alpha'(T') + 1) + \frac{1}{2}\mathrm{pc}(T') = \frac{n'}{2} + 1 = \frac{n}{2}$. This establishes Property (c).

Case 3. (T, S) can be obtained from (T', S') by operation $\mathcal{O}_3$. Let v be the attacher in T' and let $u_1u_2u_3$ be the path added to T' and vu_2 the added edge. Then, $\mathrm{sta}(u_1) = \mathrm{sta}(u_3) =$

B and $\mathrm{sta}(u_2) = A$. Further, $\alpha'(T') = \alpha'(T) - 1$.

We show firstly that Property (a) holds. Let $w \in S_A$. Let M be an arbitrary maximum matching in T and let M' be the restriction of M to the tree T'. Suppose, to the contrary, that w is M-unmatched. If $w = u_2$, then by the maximality of M, the vertex w is M-matched, a contradiction. Hence, $w \in V(T')$. If vu_2, then M' has size $|M| - 1 = \alpha'(T) - 1$. Since $\alpha'(T') = \alpha'(T) - 1$, the matching M' is a maximum matching in T'. Applying the inductive hypothesis to the labeled tree $(T', S') \in \mathcal{T}$ the vertex w is M'-matched in T' and therefore M-matched in T, a contradiction. Therefore, $vu_2 \notin M$, implying that $u_1u_2 \in M$ or $u_2u_3 \in M$. Renaming u_1 and u_3, if necessary, we may assume that $u_1u_2 \in M$. Thus, $M' = M \setminus \{u_1u_2\}$. Once again, M' has size $\alpha'(T) - 1$ and is a maximum matching in T'. Applying the inductive hypothesis to the labeled tree $(T', S') \in \mathcal{T}$ the vertex w is M'-matched in T' and therefore M-matched in T, a contradiction. This establishes Property (a).

We show secondly that Property (b) holds. Let $w \in S_B \cup S_C$. Suppose that $w = u_1$. Let M^* be a maximum matching of T'. Then, the matching $M_w = M^* \cup \{u_2u_3\}$ is a maximum matching in T such that w is M_w-unmatched. Analogously, if $w = u_2$, then there exists a maximum matching M_w in T such that w is M_w-unmatched. Hence we may assume that $w \in V(T')$, for otherwise the desired result holds. Applying the inductive hypothesis to the labeled tree $(T', S') \in \mathcal{T}$, there is a maximum matching M^* of T' such that the vertex w is M^*-unmatched. Thus the matching $M_w = M^* \cup \{u_1u_2\}$ is a maximum matching in T such that w is M_w-unmatched. This establishes Property (b).

We prove next that Property (c) holds. As observed earlier, $\alpha'(T') = \alpha'(T) - 1$. In this case, we note that $\mathrm{pc}(T) = \mathrm{pc}(T') + 1$. Therefore, $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = (\alpha'(T') + 1) + \frac{1}{2}(\mathrm{pc}(T') + 1) = \frac{n'}{2} + \frac{3}{2} = \frac{n}{2}$. This establishes Property (c).

Case 4. (T, S) *can be obtained from* (T', S') *by operation* $\mathcal{O}_4$. Let v be the attacher in T' and let $u_1u_2u_3u_4u_5$ be the path added to T' and vu_3 the added edge. Then, $\mathrm{sta}(u_1) = \mathrm{sta}(u_5) = B$, $\mathrm{sta}(u_2) = \mathrm{sta}(u_4) = A$, and $\mathrm{sta}(u_3) = C$.

We show that $\alpha'(T') = \alpha'(T) - 2$. Since M is a maximum matching in T, both vertices u_2 and u_4 are M-matched. If $vu_3 \notin M$, then $\alpha'(T') \geq |M'| = |M| - 2 = \alpha'(T) - 2$. If $vu_3 \in M$, then $M' = M \setminus \{u_1u_2, u_4u_5, vu_3\}$ and $|M'| = |M| - 3$. Applying the inductive hypothesis to the labeled tree $(T', S') \in \mathcal{T}$, the matching M' is not a maximum matching in T' since the vertex $v \in S'_A$ is M'-unmatched. Hence in this case, $\alpha'(T') \geq |M'| + 1 = |M| - 2 = \alpha'(T) - 2$. In both cases, $\alpha'(T') \geq \alpha'(T) - 2$. Conversely, every matching in T' can be extended to a matching in T by adding to it the edges u_1u_2 and u_4u_5, and so $\alpha'(T) \geq \alpha'(T') + 2$. Consequently, $\alpha'(T') = \alpha'(T) - 2$.

We show firstly that Property (a) holds. Let $w \in S_A$. Let M be an arbitrary maximum matching in T and let M' be the restriction of M to the tree T'. Suppose, to the contrary, that w is M-unmatched. If $w = u_2$ or if $w = u_4$, then by the maximality of M, the vertex w is M-matched, a contradiction. Hence, $w \in V(T')$. If $vu_3 \notin M$, then M' has size $|M| - 2 = \alpha'(T) - 2$. Since $\alpha'(T') = \alpha'(T) - 2$, the matching M' is a maximum matching in T'. Applying the inductive hypothesis to the labeled tree $(T', S') \in \mathcal{T}$, the

vertex w is M'-matched in T' and therefore M-matched in T, a contradiction. Therefore, $vu_3 \in M$, implying that $M' = M \setminus \{u_1u_2, u_4u_5, vu_3\}$ and $|M'| = |M| - 3$.

We now consider the matching M'. Let P_v be a longest path in T' starting at v and whose edges are alternately not in M' and in M' and whose vertices are alternately in S'_A and in S'_C. Let P_v be a (v, z)-path. By Observation 5, every vertex in the set S'_A has at least two neighbors in $S'_B \cup S'_C$. We now proceed analogously as in the proof of Case 1 presented earlier.

If $v = z$, then the vertex v has a leaf-neighbor u' that belongs to S'_B and is M-unmatched. We now consider the matching $M^* = M' \cup \{u'v\}$. Since M^* has size $|M'| + 1 = |M| - 2 = \alpha'(T) - 2 = \alpha'(T')$, the matching M^* is a maximum matching in T'. Since $w \notin \{u', v\}$ and w is M-unmatched, the vertex w is M^*-unmatched, contradicting Property (a). Hence, $v \neq z$.

If $z \in S'_C$, then P_v is an M'-augmenting (v, z)-path in T'. We now consider the matching $M^* = M \triangle E(P_v)$. Since M^* has size $|M'| + 1 = |M| - 2 = \alpha'(T) - 2 = \alpha'(T')$, the matching M^* is a maximum matching in T'. Since $w \notin V(P_v)$ and w is M-unmatched, the vertex w is M^*-unmatched, contradicting Property (a). Hence, $z \in S'_A$.

Since $z \in S'_A$, the vertex z has a leaf-neighbor u' that belongs to S'_B and is M-unmatched. We now extend the path P_v by adding to it the vertex z' and the edge zz' to produce a new path P which is an M'-augmenting (v, z')-path in T'. We now consider the matching $M^* = M \triangle E(P)$. Since M^* has size $|M'| + 1 = |M| - 2 = \alpha'(T) - 2 = \alpha'(T')$, the matching M^* is a maximum matching in T'. Since $w \notin V(P)$ and w is M-unmatched, the vertex w is M^*-unmatched, contradicting Property (a). Therefore, w is M-matched. Since w was chosen to be an arbitrary vertex in S_A, this proves that Property (a) holds.

We show secondly that Property (b) holds. Let $w \in S_B \cup S_C$. Suppose that $w \notin V(T')$, and so $w \in \{u_1, u_3, u_5\}$. Let M^* be a maximum matching of T'. If $w = u_1$, let $M_w = M^* \cup \{u_2u_3, u_4u_5\}$. If $w = u_3$, let $M_w = M^* \cup \{u_1u_2, u_4u_5\}$. If $w = u_5$, let $M_w = M^* \cup \{u_1u_2, u_3u_4\}$. In all cases, the matching M_w is a maximum matching in T such that w is M_w-unmatched. Hence we may assume that $w \in V(T')$, for otherwise the desired result holds. Applying the inductive hypothesis to the labeled tree $(T', S') \in \mathcal{T}$, there is a maximum matching M^* of T' such that the vertex w is M^*-unmatched. Thus the matching $M_w = M^* \cup \{u_1u_2, u_4u_5\}$ is a maximum matching in T such that w is M_w-unmatched. This establishes Property (b).

We prove next that Property (c) holds. As observed earlier, $\alpha'(T') = \alpha'(T) - 2$. We note that here $\mathrm{pc}(T) = \mathrm{pc}(T') + 1$. Therefore, $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = (\alpha'(T') + 2) + \frac{1}{2}(\mathrm{pc}(T') + 1) = \frac{n'}{2} + \frac{5}{2} = \frac{n}{2}$. This establishes Property (c) and completes the proof of Lemma 11. $\Box$

Lemma 12 *If T is a tree of order $n \geq 3$ satisfying $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = \frac{n}{2}$, then $(T, S) \in \mathcal{T}$ for some labeling S.*

Proof. We proceed by induction on the order $n \geq 3$ of a tree T satisfying $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = \frac{n}{2}$. If $n = 3$, then $T = P_3$. Letting $S = S_0^*$, we note that $(T, S) \in \mathcal{T}$. This establishes the

base case. Let $n \geq 4$ and assume that if T' is a tree of order n', where $3 \leq n' < n$, satisfying $\alpha'(T') + \frac{1}{2}\mathrm{pc}(T') = \frac{n'}{2}$, then $(T', S') \in \mathcal{T}$ for some labeling S'. Let T be a tree of order n satisfying $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = \frac{n}{2}$.

If T is a star, then let S be the labeling that assigns status A to the central vertex of the star and status B to every leaf. Then, (T, S) can be obtained from the labeled tree $(P_3, S_0^*) \in \mathcal{T}$ by repeated applications of Operation $\mathcal{O}_1$. Thus, $(T, S) \in \mathcal{T}$. Hence we may assume that $\mathrm{diam}(T) \geq 3$, for otherwise the desired result follows.

Let P be a longest path in T and suppose that P is an (r, u)-path. Necessarily, r and u are leaves in T. We now root the tree T at the vertex r. Let v be the parent of u, and let w be the parent of v in the rooted tree T. Since $\mathrm{diam}(T) \geq 3$, we note that $w \neq r$, implying that $d_T(w) \geq 2$. Let x be the parent of w.

Suppose that $d_T(v) \geq 4$. Let $T' = T - u$ and let T' have order n', and so $n' = n - 1$. Then, $\alpha'(T) = \alpha'(T')$ and $\mathrm{pc}(T) = \mathrm{pc}(T') + 1$. Applying Lemma 10 to the tree T', we see that

$$\frac{n}{2} = \alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = \alpha'(T') + \frac{1}{2}(\mathrm{pc}(T') + 1) \geq \frac{n'}{2} + \frac{1}{2} = \frac{n}{2}.$$

Hence, we must have equality throughout the above inequality chain. In particular, $\alpha'(T') + \frac{1}{2}\mathrm{pc}(T') = \frac{n'}{2}$. Applying the inductive hypothesis to the tree T', we note that $(T', S') \in \mathcal{T}$ for some labeling S'. Let S be the labeling obtained from S' by assigning to the vertex u status B. Since v is a support vertex in T', the vertex $v \in S'_A$ by Observation 5(a). Therefore, the labeled tree (T, S) can be obtained by the labeled tree (T', S') by applying Operation $\mathcal{O}_1$, implying that $(T, S) \in \mathcal{T}$. Therefore, we may assume that $d_T(v) \leq 3$, for otherwise the desired result follows.

Suppose that $d_T(v) = 3$. Let u_1 and u_2 denote the two children of v, where $u = u_1$. We now let T' be the tree obtained from T by deleting u_1, u_2 and v; that is, $T' = T - D[v]$. Let T' have order n', and so $n' = n - 3$. Then, $\alpha'(T) = \alpha'(T') + 1$ and $\mathrm{pc}(T) = \mathrm{pc}(T') + 1$. Applying Lemma 10 to the tree T', we see that

$$\frac{n}{2} = \alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = (\alpha'(T') + 1) + \frac{1}{2}(\mathrm{pc}(T') + 1) \geq \frac{n'}{2} + \frac{3}{2} = \frac{n}{2}.$$

Hence, we must have equality throughout the above inequality chain. In particular, $\alpha'(T') + \frac{1}{2}\mathrm{pc}(T') = \frac{n'}{2}$. Since $\{w, x\} \subseteq V(T')$, we note that $n' \geq 2$. If $n' = 2$, then the tree T is determined (and is obtained from a star $K_{1,3}$ by subdividing one edge once). In this case, $n = 5$, $\alpha'(T) = 2$ and $\mathrm{pc}(T) = 2$, implying that $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) > \frac{n}{2}$, a contradiction. Therefore, $n' \geq 3$. Applying the inductive hypothesis to the tree T', we note that $(T', S') \in \mathcal{T}$ for some labeling S'. Let S be the labeling obtained from S' by assigning status A to the vertex v and status B to both u_1 and u_2. Further, if w has status B in the labeling S' (that is, if $w \in S'_B$), then change the status of v from B to C in the labeling S (that is, $w \in S_C$). Since the labeled tree (T, S) can be obtained from the labeled tree (T', S') by applying Operation $\mathcal{O}_3$, we therefore have that $(T, S) \in \mathcal{T}$. Hence we may assume that $d_T(v) = 2$, for otherwise the desired result follows. Analogously, we may assume that every child of w that is not a leaf has degree 2 in T.

Suppose that $d_T(w) \ge 3$. Let w have k children, and so $k = d_T(w) - 1 \ge 2$. By our earlier assumptions, every child of w is a leaf or is a support vertex of degree 2 in T. Let w have k_1 children that are non-leaves and k_2 children that are leaves. Thus, $k = k_1 + k_2$, $k \ge 2$, $k_1 \ge 1$, and $k_2 \ge 0$. Suppose that $k \ge 3$. Let $T' = T - \{u, v\}$ be obtained from T by deleting the vertices u and v, and let T' have order n'. Thus, $n' = n - 2$. Then, $\alpha'(T) = \alpha'(T') + 1$ and $\mathrm{pc}(T) = \mathrm{pc}(T') + 1$. Applying Lemma 10 to the tree T', we see that

$$\frac{n}{2} = \alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = (\alpha'(T') + 1) + \frac{1}{2}(\mathrm{pc}(T') + 1) \ge \frac{n'}{2} + \frac{3}{2} > \frac{n}{2},$$

a contradiction. Therefore, $k = 2$. Suppose that $k_1 = 1$, and so $k_2 = 1$. Let v' be the leaf-neighbor of w. Let $T' = T - D[w]$ be obtained from T by deleting the vertices u, v, v' and w, and let T' have order n'. Thus, $n' = n - 4$. Then, $\alpha'(T) = \alpha'(T') + 2$ and $\mathrm{pc}(T) = \mathrm{pc}(T') + 1$. Applying Lemma 10 to the tree T', we see that

$$\frac{n}{2} = \alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = (\alpha'(T') + 2) + \frac{1}{2}(\mathrm{pc}(T') + 1) \ge \frac{n'}{2} + \frac{5}{2} > \frac{n}{2},$$

a contradiction. Therefore, $k = k_1 = 2$. Let v' be the child of w different from v and let u' be the child of v'. We now let T' be the tree obtained from T by deleting u, u', v, v' and w; that is, $T' = T - D[w]$. Let T' have order n', and so $n' = n - 5$. Then, $\alpha'(T) \ge \alpha'(T') + 2$ and $\mathrm{pc}(T) = \mathrm{pc}(T') + 1$. Applying Lemma 10 to the tree T', we see that

$$\frac{n}{2} = \alpha'(T) + \frac{1}{2}\mathrm{pc}(T) \ge (\alpha'(T') + 2) + \frac{1}{2}(\mathrm{pc}(T') + 1) \ge \frac{n'}{2} + \frac{5}{2} = \frac{n}{2}.$$

Hence, we must have equality throughout the above inequality chain. In particular, $\alpha'(T) = \alpha'(T') + 2$ and $\alpha'(T') + \frac{1}{2}\mathrm{pc}(T') = \frac{n'}{2}$. Since P is a longest path in T, we note that $n' \ge 2$. If $n' = 2$, then the tree T is determined (and is obtained from a star $K_{1,3}$ by subdividing every edge once). In this case, $n = 7$, $\alpha'(T) = 3$ and $\mathrm{pc}(T) = 2$, implying that $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) > \frac{n}{2}$, a contradiction. Therefore, $n' \ge 3$. Applying the inductive hypothesis to the tree T', we note that $(T', S') \in \mathcal{T}$ for some labeling S'. Recall that x is the parent of w in the rooted tree T. If $x \in S'_B \cup S'_C$, then by Lemma 11, there exists a maximum matching M' in T' such that w is M'-unmatched. In this case, $M' \cup \{xw, uv, u'v'\}$ is a matching in T of size $|M'| + 3 = \alpha'(T') + 3 > \alpha'(T)$, a contradiction. Therefore, $x \in S'_A$. Let S be the labeling obtained from S' by assigning status C to the vertex w, status A to both v and v', and status B to both u and u'. Since $x \in S'_A$, the labeled tree (T, S) can be obtained by the labeled tree (T', S') by applying Operation $\mathcal{O}_4$, implying that $(T, S) \in \mathcal{T}$. Therefore, we may assume that $d_T(w) = 2$, for otherwise the desired result follows.

We now let T' be the tree obtained from T by deleting u and v; that is, $T' = T - \{u, v\}$. Let T' have order n', and so $n' = n - 2$. Then, $\alpha'(T) = \alpha'(T') + 1$. Since $d_T(w) = 2$, we note that w is a leaf in T' and $\mathrm{pc}(T) = \mathrm{pc}(T')$. Applying Lemma 10 to the tree T', we see that

$$\frac{n}{2} = \alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = (\alpha'(T') + 1) + \frac{1}{2}\mathrm{pc}(T') \ge \frac{n'}{2} + 1 = \frac{n}{2}.$$

Hence, we must have equality throughout the above inequality chain. In particular, $\alpha'(T') + \frac{1}{2}\mathrm{pc}(T') = \frac{n'}{2}$. If $n' = 2$, then the tree T is determined and $T = P_4$. In this case,

$n = 4$, $\alpha'(T) = 2$ and $\mathrm{pc}(T) = 1$, implying that $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) > \frac{n}{2}$, a contradiction. Therefore, $n' \ge 3$. Applying the inductive hypothesis to the tree T', we note that $(T', S') \in \mathcal{T}$ for some labeling S'. As observed earlier, w is a leaf in T'. By Observation 5(b), the vertex $w \in S'_B$. Let S be the labeling obtained from S' by assigning status A to the vertex v, status B to the vertex u and reassigning the status of vertex w to be status C. Since $w \in S'_B$, the labeled tree (T, S) can be obtained by the labeled tree (T', S') by applying Operation $\mathcal{O}_2$, implying that $(T, S) \in \mathcal{T}$. This completes the proof of Lemma **??**. $\Box$

As an immediate consequence of Lemmas 10, 11 and 12, we have the result of Theorem 6. This result implies that given any graph G of order n, for every spanning tree T of G we know that $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) \ge \frac{n}{2}$. We are now in a position to prove the same lower bound holds for G as well. Recall the statement of Theorem 7.

Theorem 7 *If G is a graph of order n, then $\alpha'(G) + \frac{1}{2}\mathrm{pc}(G) \ge \frac{n}{2}$. Further for $n \ge 3$, $\alpha'(G) + \frac{1}{2}\mathrm{pc}(G) = \frac{n}{2}$ if and only if G has a spanning tree T such that*

(a) *$(T, S) \in \mathcal{T}$ for some labeling S.*
(b) *$\alpha'(G) = \alpha'(T)$.*
(c) *$\mathrm{pc}(G) = \mathrm{pc}(T)$.*

Proof. By linearity, it suffices for us to restrict our attention to connected graphs. Let G be a connected graph of order n, and let $\mathcal{P}$ be a minimum path cover in G, implying that $|\mathcal{P}| = \mathrm{pc}(G)$. Let T be a spanning tree obtained from the disjoint union of the $\mathrm{pc}(G)$ paths that belong to the path cover $\mathcal{P}$ by adding $\mathrm{pc}(G) - 1$ edges (in such a way that the resulting graph is connected). Since $\mathcal{P}$ is a path cover in T, we note that $\mathrm{pc}(T) \le |\mathcal{P}| = \mathrm{pc}(G)$. Since adding edges to a tree cannot increase its path covering number, we note that $\mathrm{pc}(G) \le \mathrm{pc}(T)$. Consequently, $\mathrm{pc}(G) = \mathrm{pc}(T)$. Every matching in T is a matching in G, implying that $\alpha'(G) \ge \alpha'(T)$. Hence, applying Lemma 10 to the tree T, we see that

$$\alpha'(G) + \frac{1}{2}\mathrm{pc}(G) \ge \alpha'(T) + \frac{1}{2}\mathrm{pc}(T) \ge \frac{n}{2}.$$

Suppose that $n \ge 3$ and $\alpha'(G) + \frac{1}{2}\mathrm{pc}(G) = \frac{n}{2}$. Then, we must have equality throughout the above inequality chain, implying that $\alpha'(G) = \alpha'(T)$ and $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = \frac{n}{2}$. By Lemma 12, $(T, S) \in \mathcal{T}$ for some labeling S. Thus, conditions (a), (b) and (c) in the statement of the theorem all hold. Conversely, if conditions (a), (b) and (c) in the statement of the theorem all hold, then by Lemma 11, $\alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = \frac{n}{2}$, implying that

$$\alpha'(G) + \frac{1}{2}\mathrm{pc}(G) = \alpha'(T) + \frac{1}{2}\mathrm{pc}(T) = \frac{n}{2}.$$

This completes the proof of Theorem 7. $\Box$

4 Applications to Domination Parameters

We remark that the upper bound of Theorem 3 on the total domination number in terms of its matching and path covering numbers is sharp even for graphs with minimum degree 2.

For example, let F be a star $K_{1,k-1}$ on $k \ge 3$ vertices and let G be obtained from F as follows: For each vertex v of F, add a 6-cycle and join v to one vertex of this cycle. The resulting graph G has order $7k$, $\gamma_t(G) = 4k$, $\alpha'(G) = 3k + 1$ and $\mathrm{pc}(G) = k - 1$, implying that $\gamma_t(G) = 4k = \alpha'(G) + \mathrm{pc}(G)$.

As an application of Theorem 7, we show, however, that the bound of Theorem 3 can be improved considerably if we restrict the minimum degree of the graph to be at least three. If G is a graph of order n with minimum degree at least 3, then it is well-known (see, for example, [1, 20]) that $\gamma_t(G) \le \frac{n}{2}$. Further the graphs achieving equality in this bound are characterized in [17]. We observe that these extremal graphs in [17] do not achieve equality in the bound of Theorem 7. Therefore as a consequence of these results on total domination in graphs with minimum degree at least 3, the following improvement of Theorem 3 follows readily from Theorem 7: If G is a graph with $\delta(G) \ge 3$, then $\gamma_t(G) < \alpha'(G) + \frac{1}{2}\mathrm{pc}(G)$. We remark that there are two (infinite) families of connected cubic graphs G satisfying $\gamma_t(G) = \frac{n}{2}$ (as shown in [17]). Each such graph G satisfies $\alpha'(G) + \frac{1}{2}(\mathrm{pc}(G) - 1) = \frac{n}{2}$. Consequently, there are infinitely many connected cubic graphs G satisfying $\gamma_t(G) = \alpha'(G) + \frac{1}{2}(\mathrm{pc}(G) - 1)$. This establishes the result of Corollary 8. Recall its statement.

Corollary 8 *If G is a graph with $\delta(G) \ge 3$, then*

$$\gamma_t(G) \le \alpha'(G) + \frac{1}{2}(\mathrm{pc}(G) - 1),$$

and this bound is tight.

As observed earlier, the relationship between the neighborhood total domination number and the matching number of a graph behaves quite differently from the relationship between the domination number and the matching number of a graph. Certainly there exist many graphs G for which $\alpha'(G) \ge \gamma_{\mathrm{nt}}(G)$. For example, if G is obtained from a connected graph H by adding at least one pendant edges to each vertex of H, then $\alpha'(G) = \gamma_{\mathrm{nt}}(G) = n(H)$ (noting that the set $V(H)$ is a minimum NTD-set in G). If $G = K_{n,n}$ where $n \ge 3$, then $\alpha'(G) = n > 2 = \gamma_{\mathrm{nt}}(G)$. What is not quite as obvious is that there exist an infinite number of graphs for which $\alpha'(G) < \gamma_{\mathrm{nt}}(G)$, even for arbitrary large (but fixed) minimum degree.

Theorem 13 *For every integer $\delta \ge 1$, there exists a graph G with $\delta(G) = \delta$ satisfying $\gamma_{\mathrm{nt}}(G) > \alpha'(G)$.*

Proof. When $\delta = 1$, take G to be a star on at least two vertices, while if $\delta = 2$, take G to be a 5-cycle. Hence we may assume that $\delta \ge 3$. Let $n = \delta(\delta + 1)$, let A be a set of n elements, and let B be the set of all δ-element subsets of A. Viewing the elements of A and B as vertices, let $G = G_n^\delta$ be the bipartite graph formed by taking A as one partite set and B as the other partite set where vertex v_a corresponding to element $a \in A$ is adjacent to vertex v_b corresponding to the δ-element subset $b \in B$ if and only if a is contained in b. Thus, every vertex in B has degree δ, while every vertex in A has degree $\binom{n-1}{\delta-1}$. Furthermore, G is a bipartite graph with minimum degree δ and order $n + \binom{n}{\delta}$. Now, $\alpha'(G) \le \min(|A|, |B|) = |A| = n$. It is easy to find a matching of size n in G, and so $\alpha'(G) = n$.

We show next that $\gamma_{\rm nt}(G) \ge n+1$. Let D be an arbitrary NTD-set in G and write $D_A = D \cap A$ and $D_B = D \cap B$. It follows that $|D| = |D_A| + |D_B|$. We show that $|D_B| \ge \delta + 1$. Let S be an arbitrary δ-element subset of A and let v_S be the vertex in B associated with S. Suppose that D_B contains no neighbor of a vertex in S; that is, $D_B \cap N(S) = \emptyset$. On the one hand, if S contains a vertex of $A \setminus D_A$, then such a vertex would not be dominated by D. On the other hand, if $S \subseteq D_A$, then the vertex v_S would be isolated in the subgraph induced by $N(D)$. Both cases produce a contradiction. Therefore, D_B contains at least one vertex in $N(S)$. This is true for any δ-element subset, S, of A. In particular, this implies that $|D_B| \ge \lfloor |A|/\delta \rfloor = \lfloor n/\delta \rfloor = \delta + 1$ since each vertex of B dominates exactly δ vertices in A and $|A| = \delta^2 + \delta$.

We are now in a position to show that $|D| \ge n+1$. If $|D_A| = n-k$ for some k where $k < \delta$, then $|D| = |D_A| + |D_B| > (n-\delta) + (\delta+1) = n+1$. Hence, we may assume that $|D_A| = n-k$ for some k where $k \ge \delta$, for otherwise $|D| \ge n+1$, as desired. In order to dominate the $\binom{k}{\delta}$ vertices in B associated with δ-element subsets of $A \setminus D_A$, all these $\binom{k}{\delta}$ vertices belong to the set D_B. Further, since D_B contains at least one vertex in $N(S)$ for every δ-element subset S of A, the set D_B contains at least $\lfloor |D_A|/\delta \rfloor = \lfloor (n-k)/\delta \rfloor$ additional vertices, implying that

$$|D| \ge n - k + \binom{k}{\delta} + \left\lfloor \frac{n-k}{\delta} \right\rfloor. \tag{1}$$

Let $f(k)$ denote the function on the right hand side of Inequality (1). It suffices for us to show that $f(k) \ge n+1$. If $k = \delta$, then $f(k) = n+1$. Hence we may assume that $k \ge \delta+1$. Recall that $n = \delta(\delta+1)$ and, by assumption, $\delta \ge 3$. If $D_A = \emptyset$, then $k = n = \delta(\delta+1)$ and $f(k) = \binom{k}{\delta} = \binom{\delta(\delta+1)}{\delta} > n+1$. Suppose that $D_A \ne \emptyset$, and so $n-k \ge 1$. If $n-k < \delta$, then $k \ge \delta^2+1$ and $f(k) \ge 1 + \binom{\delta^2+1}{\delta} \ge n+1$. If $n-k \ge \delta$ and $k \ge \delta+1$, then $\lfloor \frac{n-k}{\delta} \rfloor \ge 1$ and $\binom{k}{\delta} \ge k$, implying that $f(k) \ge (n-k)+k+1 = n+1$. In all cases, we have shown that $f(k) \ge n+1$, as claimed. Therefore, if $k \ge \delta$, then $|D| \ge f(k) \ge n+1$. Since D is an arbitrary NTD-set of G and $|D| \ge n+1$, we deduce that $\gamma_{\rm nt}(G) \ge n+1 > \alpha'(G)$, as desired. $\Box$

Although we have seen that, given any graph G, $\gamma_{\rm nt}(G)$ may be larger than $\alpha'(G)$, it turns out that we can upper bound $\gamma_{\rm nt}(G)$ in terms of the matching number and the path covering number of G. As a consequence of Theorem 4 and Theorem 7, we have a proof of Corollary 9. Recall its statement.

Corollary 9 *If G is a connected graph on at least three vertices, then*

$$\gamma_{\rm nt}(G) \le \alpha'(G) + \frac{1}{2}{\rm pc}(G)$$

unless $G \in \{P_3, P_5, C_5\}$ in which case $\gamma_{\rm nt}(G) = \alpha'(G) + \frac{1}{2}({\rm pc}(G)+1)$.

Proof. Let G be a connected graph of order $n \ge 3$. If $G = C_5$ or if G is a subdivided star, then by Theorem 4, $\gamma_{\rm nt}(G) = (n+1)/2$. If G is a subdivided star and $n \ge 7$, then $\alpha'(G) = (n-1)/2$ and ${\rm pc}(G) \ge 2$, and so $\alpha'(G) + \frac{1}{2}{\rm pc}(G) \ge (n+1)/2 = \gamma_{\rm nt}(G)$. If G is a

subdivided star and $n < 7$, then $G \in \{P_3, P_5\}$ and $\alpha'(G)+\frac{1}{2}(\mathrm{pc}(G)+1) \geq (n+1)/2 = \gamma_{\mathrm{nt}}(G)$. If $G = C_5$, then once again $\alpha'(G) + \frac{1}{2}(\mathrm{pc}(G) + 1) \geq (n + 1)/2 = \gamma_{\mathrm{nt}}(G)$. Therefore, we may assume that G is neither a 5-cycle nor a subdivided star, for otherwise the desired result follows. Hence by Theorem 4, $\gamma_{\mathrm{nt}}(G) \leq \frac{n}{2}$ and the desired result follows by Theorem 7. $\Box$

We remark that the bound of Corollary 9 is tight as illustrated by the following example. Let H be an arbitrary hamiltonian graph of odd order $k \geq 1$ and let G be the graph obtained from H by the following operation: for each vertex $x \in V(H)$, add a new path P_3 and an edge joining its central vertex to x. Equivalently, for $k \geq 1$ odd, the graph G is obtained from the disjoint union of $k \geq 1$ stars $K_{1,3}$ by selecting one leaf from each star and adding any number of edges between the selected k leaves so that they induce a graph that contains a hamiltonian path. The resulting graph G has order $4k$, $\gamma_{\mathrm{nt}}(G) = 2k$, $\alpha'(G) = k+(k-1)/2$ and $\mathrm{pc}(G) = k + 1$, implying that $\gamma_{\mathrm{nt}}(G) = 2k = \alpha'(G) + \frac{1}{2}\mathrm{pc}(G)$.

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